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Absolute Continuity of Multivariate Distributions of Class L

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It is shown that every genuinely d-dimensional distribution of class L on R^d is absolutely continuous. This extends the known fact in one dimension to all finite dimensions.

1. Introduction and Result

A probability distribution μ on R^d is said to be of class L, if there are sequences $\{X_n\}$, $\{a_n\}$, and $\{b_n\}$ of independent R^d-valued random variables, d-vectors, and positive numbers, respectively, such that the distribution of $b_n^{-1}\sum_{j=1}^n X_j - a_n$ converges (weakly) to μ and $\max_{1\leqslant j\leqslant n} P(b_n^{-1}|X_j| > \varepsilon)\to 0$ as $n\to\infty$ for any $\varepsilon > 0$. A measure on R^d is said to be genuinely d-dimensional if it is not concentrated on any $(d-1)$-dimensional hyperplane. In this paper we will prove the following theorem.

THEOREM. *If μ is a genuinely d-dimensional distribution of class L on R^d, then μ is absolutely continuous (with respect to Lebesgue measure).*

In case $d=1$, the result is well-known [1, 7]. In case $d\geqslant 2$, we have discussed continuity properties of distributions of class L in Ref. [2]. But, in Ref. [2], we have not obtained a necessary and sufficient condition for absolute continuity. There exists a genuinely d-dimensional distribution μ of class L such that its characteristic function $\hat{\mu}(z)$ is not square-integrable (for example, consider a product measure of gamma distributions with appropritate parameters). Hence the absolute continuity does not seem to be obtainable from consideration of the behavior of $\hat{\mu}(z)$ as $|z|\to\infty$. Our method of proof of the above theorem is to divide the consideration into two

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cases—the case where every proper subspace has Lévy measure zero and the other case. The first case will be treated by proving two lemmas, while the second case will be discussed by induction.

2. Proof

For x, $z \in R^d$, we denote the inner product of x and z by xz and the Euclidean norm of x by $|x|$. Let $S = \{x \in R^d : |x| = 1\}$, the unit sphere in R^d, and let $R_+ = (0, \infty)$, the open half line. For $E \subset R_+$ and $B \subset S$, we denote by EB the set of points x such that $x = u\xi$, $u \in E$, $\xi \in B$. The class of Borel subsets of a set T is denoted by $\mathscr{B}(T)$ in general. If μ is an infinitely divisible distribution on R^d, then its characteristic function $\hat{\mu}(z)$, $z \in R^d$, is uniquely represented in the form

$$\hat{\mu}(z) = \exp\left\{i\gamma z - A(z) + \int_{R^d} \left(e^{izx} - 1 - \frac{izx}{1+|x|^2}\right) \nu(dx)\right\}, \tag{2.1}$$

where $\gamma \in R^d$, $A(z)$ is a nonnegative quadratic form, and ν is a measure (called *Lévy measure*) on R^d such that $\nu(\{0\}) = 0$ and

$$\int_{R^d} \frac{|x|^2}{1+|x|^2} \nu(dx) < \infty.$$

The characteristic functions of multivariate distributions of class L are studied in Refs. [2, 5, 6]. A probability distribution μ on R^d is of class L if and only if it is infinitely divisible and its Lévy measure ν is either identically zero or of the form

$$\nu(EB) = \int_B \lambda(d\xi) \int_E k_\xi(u)\, u^{-1}\, du \qquad \text{for} \quad E \in \mathscr{B}(R_+), B \in \mathscr{B}(S), \tag{2.2}$$

where λ is a probability measure on S, and $k_\xi(u)$ is nonnegative, nonincreasing in u and Borel measurable in ξ, and

$$0 < \int_0^\infty k_\xi(u)\, u(1+u^2)^{-1}\, du = c < \infty$$

with c independent of ξ [2, p. 213]. The representation is unique. The measure λ is called the spherical component of the Lévy measure of μ and $k_\xi(u)$ is called the k-function of μ.

The following lemma extends a result of Tucker [4], Fisz and Varadarajan [1], and Zolotarev [7]. Define $\tilde{\nu}$ by $\tilde{\nu}(dx) = |x|^2(1+|x|^2)^{-1}\, \nu(dx)$.

LEMMA 1. *Let μ be an infinitely divisible distribution on R^d. If its Lévy measure ν has infinite total mass and if, for some n, the n-fold convolution $\tilde{\nu}^{n*}$ of $\tilde{\nu}$ is absolutely continuous, then μ is absolutely continuous.*

Proof. For each $k > 0$, let μ_k be a compound Poisson distribution on R^d with characteristic function

$$\hat{\mu}_k(z) = \exp\left\{\int_{|x|>k^{-1}} (e^{ixz} - 1)\, \nu(dx)\right\} = \exp\left\{c_k \int_{R^d} (e^{ixz} - 1)\, \nu_k(dx)\right\},$$

where

$$c_k = \int_{|x|>k^{-1}} \nu(dx), \qquad \nu_k(dx) = c_k^{-1} \chi_{\{|x|>k^{-1}\}}(x)\, \nu(dx).$$

The letter χ denotes indicator function. This μ_k is a factor of μ, that is, $\mu = \mu_k * \mu_{k0}$ with some μ_{k0}. We have

$$\mu_k = \sum_{l=0}^{n-1} e^{-c_k} \frac{c_k^l}{l!} \nu_k^{l*} + \sum_{l=n}^{\infty} e^{-c_k} \frac{c_k^l}{l!} \nu_k^{l*}.$$

Denote the first term by $a_{k1}\mu_{k1}$ and the second term by $a_{k2}\mu_{k2}$, where μ_{k1} and μ_{k2} are probability distributions. Since $c_k \to \infty$ by our assumption, we have $a_{k1} \to 0$ as $k \to \infty$. It is easy to see that our assumption of the abosolute continuity of $\tilde{\nu}^{n*}$ implies the absolute continuity of μ_{k2}. Let $\mu = b_1\mu_{(1)} + b_2\mu_{(2)}$ be the Lebesgue decomposition of μ, where $\mu_{(1)}$ is a singular distribution and $\mu_{(2)}$ is an absolutely continuous distribution. Since $\mu = a_{k1}\mu_{k1} * \mu_{k0} + a_{k2}\mu_{k2} * \mu_{k0}$ and the second term is absolutely continuous, we get $b_1 \leqslant a_{k1}$. It follows that $b_1 = 0$, and the proof is complete.

LEMMA 2. *Let ν be the Lévy measure of a distribution of class L on R^d. Suppose that $\nu(H) = 0$ for every proper subspace H of R^d. Then, the d-fold convolution $\tilde{\nu}^{d*}$ of $\tilde{\nu}$ is absolutely continuous.*

Proof. If ν is identically zero, the assertion is trivial. Suppose that ν is not identically zero and let λ be the spherical component of ν. The assumption implies that $\lambda(S \cap H) = 0$ for every proper subspace H. Let $F \in \mathscr{B}(R^d)$ be a set of Lebesgue measure zero. We have

$$\begin{aligned}\tilde{\nu}^{d*}(F) &= \int_{(R^d)^d} \chi_F(x_1 + \cdots + x_d) \prod_{j=1}^{d} \tilde{\nu}(dx_j) \\ &= \int_{S^d} \prod_{j=1}^{d} \lambda(d\xi_j) \int_{R_+^d} \chi_F(u_1\xi_1 + \cdots + u_d\xi_d) \prod_{j=1}^{d} (k_{\xi_j}(u_j)\, u_j(1+u_j^2)^{-1}\, du_j).\end{aligned} \tag{2.3}$$

If $\xi_1,\dots,\xi_d$ are linearly independent, then

$$\int_{R_+^d} \chi_F(u_1\xi_1+\cdots+u_d\xi_d)\,du_1\cdots du_d=0$$

by the change of variables $u \mapsto v=u_1\xi_1+\cdots+u_d\xi_d$ in multiple integral. Let K_r be the set of $(\xi_1,\dots,\xi_d)\in S^d$ such that $\operatorname{rank}(\xi_1,\dots,\xi_d)=r$. Let $K_r(i_1,\dots,i_r)$ be the set of $(\xi_1,\dots,\xi_d)\in K_r$ such that $\xi_{i_1},\dots,\xi_{i_r}$ are linearly independent. We have

$$K_r=\bigcup_{(i_1,\dots,i_r)} K_r(i_1,\dots,i_r).$$

The domain of integration S^d in (2.3) can be replaced by $\bigcup_{r=1}^{d-1} K_r$. If $1\leqslant r\leqslant d-1$, then, choosing $j_0\neq i_1,\dots,i_r$ and using the assumption $\lambda(S\cap H)=0$, we get

$$\int_{K_r(i_1,\dots,i_r)}\prod_{j=1}^{d}\lambda(d\xi_j)\leqslant\int_{S^{d-1}}\lambda(S\cap H(\xi_{i_1},\dots,\xi_{i_r}))\prod_{j\neq j_0}\lambda(d\xi_j)=0.$$

Here $H(\xi_{i_1},\dots,\xi_{i_r})$ is the linear subspace spanned by $\xi_{i_1},\dots,\xi_{i_r}$. We now see that $\tilde{\nu}^{d*}(F)=0$, which completes the proof.

Proof of theorem. Let μ be a genuinely d-dimensional distribution of class L on R^d. We prove our theorem by induction in d. If $d=1$, this is a result of Lemma 1. See Refs. [1, 7] or, for a different proof, Ref. [3]. Assume that $d\geqslant 2$ and that the theorem is true for lower dimensions. The characteristic function of μ has the representation (2.1) with Lévy measure of the form (2.2). If A has rank d, then it is obvious that μ is absolutely continuous. If ν identically vanishes, then A must have rank d. If ν does not identically vanish and satisfies the condition in Lemma 2, then the absolute continuity follows from Lemmas 1 and 2. So, we assume that A does not have rank d and that $\nu(H)>0$ for some proper subspace H of R^d. Let us define a subspace H_1 as follows. If A has positive rank, then let μ_1 be the Gaussian component of μ, that is, $\hat{\mu}_1(z)=\exp(-A(z))$, and let H_1 be the support of μ_1. If $A=0$, then let H_1 be the smallest subspace that contains the support of the restriction of ν to H and let μ_1 be the distribution with characteristic function

$$\hat{\mu}_1(z)=\exp\left\{\int_{S\cap H_1}\lambda(d\xi)\int_0^\infty\left(e^{iu\xi z}-1-\frac{iu\xi z}{1+u^2}\right)\frac{k_\xi(u)}{u}\,du\right\}.$$

Let $l=\dim H_1$. We have $1\leqslant l\leqslant d-1$. Let H_2 be the orthogonal complement of H_1 in R^d ($\dim H_2=d-l$). Let T_1 and T_2 be the projection operators to H_1 and H_2, respectively. For $x\in R^d$, we denote $x_1=T_1x$ and

$x_2 = T_2 x$. Define μ_2 by $\mu = \mu_1 * \mu_2$. The distributions μ_1 and μ_2 are of class L. It is easy to see that μ_1 is supported by the subspace H_1 and the restriction of μ_1 to H_1 can be identified with a genuinely l-dimensional distribution of class L on R^l. Hence, by the induction hypothesis, μ_1 is absolutely continuous with respect to the l-dimensional Lebesgue measure dx_1 on H_1. Let $\mu_1(dx) = f(x_1)\, dx_1$. We have

$$\mu(F) = \int_{R^d} \mu_2(dy) \int_{R^l} \chi_F(x_1 + y_1, y_2) f(x_1)\, dx_1, \qquad F \in \mathscr{B}(R^d).$$

Let F be of Lebesgue measure zero. We claim that $\mu(F) = 0$. Let

$$g(y_1, y_2) = \int_{R^l} \chi_F(x_1 + y_1, y_2) f(x_1)\, dx_1,$$

which is Borel measurable in (y_1, y_2). Since

$$\int_{R^{d-l}} dy_2 \int_{R^l} \chi_F(x_1, y_2)\, dx_1 = 0,$$

there is a set $F_2 \in \mathscr{B}(R^{d-l})$ with $(d-l)$-dimensional Lebesgue measure zero such that, for every $y_2 \notin F_2$,

$$\int_{R^l} \chi_F(x_1, y_2)\, dx_1 = 0.$$

It follows that $g(y_1, y_2) = \chi_{F_2}(y_2)\, g(y_1, y_2)$. Let Y be an R^d-valued random variable with distribution μ_2, and let $Y_1 = T_1 Y$ and $Y_2 = T_2 Y$. Let $p_2(\cdot)$ be the distribution of Y_2 and $p_1(\cdot \mid y_2)$ be the conditional distribution of Y_1 given $Y_2 = y_2$. Then we have

$$\mu(F) = \int_{F_2} p_2(dy_2) \int_{R^l} g(y_1, y_2)\, p_1(dy_1 \mid y_2). \tag{2.4}$$

Since p_2 is the projection of μ_2, it is also of class L. Now we see that p_2 is genuinely $(d-l)$-dimensional. In fact, let ν_2 be the Lévy measure of μ_2. By the definition of μ_2, we see that μ_2 (and hence also p_2) has Gaussian part zero. If the Lévy measure $\nu_2(T_2^{-1}(\cdot))$ of p_2 is supported by a proper subspace H_2^0 of H_2, then ν_2 is supported by $H_2^0 \oplus H_1$, which implies that ν is supported by $H_2^0 \oplus H_1$ and contradicts the assumption that μ is genuinely d-dimensional. Hence p_2 is genuinely $(d-l)$-dimensional. Therefore, by the induction hypothesis, p_2 is absolutely continuous with respect to $(d-l)$-dimensional Lebesgue measure. It follows that $p_2(F_2) = 0$. We obtain $\mu(F) = 0$ by (2.4). The proof is complete.

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